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Evading divergences in quantum field theory

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Abstract

Explicit solution of a Green function in a non-renormalizable toy model demonstrates that Green functions of the interacting theory fall off much faster than at the tree level at large momenta. This suggests a method of calculations in quantum field theory which is free of divergences.

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Since the time Dirac quantized the Maxwell's equations, infinities have plagued quantum field theory (QFT). Over the decades they have evoked all kinds of responses. A consequence of infinite number of degrees of freedom; a deficiency of perturbation theory; a result of taking products of fields at the same space-time point etc.. Nevertheless a pragmatic view of renormalization of charges and masses has had remarkable successes. First this was with the verification of predictions of QED to awesome accuracy. Next it was with the standard model of electroweak and strong interactions, where the search for the theory itself was motivated by renormalization. The renormalization group analysis of Wilson and others relate the divergences to infinite number of scales in the theory and make connection with other areas of physics. This approach make the divergences inevitable and tackles them in a non-perturbative way. However it has not been possible to bring quantum gravity into this fold at present. Many regard this as a fundamental defect of QFT requiring a new paradigm and a new theory. Planck length is believed to play a crucial role, either as a fundamental parameter of a new theory or specifying a drastic change in the nature of space-time, requiring a new way of handling the theory.

In this Note we propose a conventional way out. We propose that the Green functions of QFT have to be handled via integral equations. We illustrate that this approach is free of divergences even in so called non- renormalization theories. Tackling QFT through integral equations is nothing new. The Schwinger- Dyson equations define QFT through an infinite set of coupled non-linear integral equations. They have also been used in many contexts in various approximations, truncations and for non-perturbative analysis. We give a different orientation to these equations here.

We blame the divergences in QFT on the insistence on perturbative expansion, in particular the presumption that the Green functions of the interacting theory have the asymptotic behaviour of those of the tree approximation (i.e. the semi-classical limit). The very definition of the Green function via an integral equation force an asymptotic behaviour which makes all integrals converge. We demonstrate this with an explicit solution in a toy model here.

Consider elastic scattering of two like particles 'A' by exchange of field quanta 'B' in the ladder approximation. For non-exceptional Euclidean (and therefore off mass shell) external momenta, all internal momenta can be safely rotated to be Euclidean. We consider a toy model where the (euclidean) propagators of 'A' and 'B' quanta are respectively $1/\sqrt{q}$ and $1/q$ respectively. Here \vec{q} is the momentum in 3-(Euclidean) dimensions and $q = |\vec{q}|$. With our choice, the 1-box diagram diverges logarithmically, the 2-box diagram is linearly divergent and the N-box diagram has a superficial degree of divergence (N-1). Thus the model is like a non- renormalization theory.

The sum of the ladder diagrams can be formally specified by an integral equation . We consider the specific case of total incoming momentum being zero. This is not a serious restriction as it doesnot alter considerations of ultraviolet divergences. We denote the net Green function excluding the external legs in this case by $I(\vec{q}, \vec{q}')$ where the incoming particles of (Euclidean) momenta \vec{q} and $-\vec{q}$ are scattered into outgoing particles of momenta \vec{q}' and $-\vec{q}'$ respectively. It satisfies the integral equation [1],

$$I(\vec{q}, \vec{q}') = \frac{1}{|\vec{q} - \vec{q}'|} + e^2 \int \frac{d^3 q''}{q''} \frac{I(\vec{q}, \vec{q}')}{|\vec{q}'' - \vec{q}'|} \quad (1)$$

A perturbation expansion in e^2 formally generates the ladder diagrams with increasing degrees of divergence. Nevertheless the integral equation, (1) has a finite and meaningful solution. The reason is clear. The presumption that the full Green function has the asymptotic behaviour of the semi classical Green function is wrong. The full Green function has an asymptotic behaviour such that integral in (1) converges. This behaviour is forced by the integral equation itself.

The integral equation Eq.1 is precisely that for the zero energy Green function of the non-relativistic Rutherford scattering problem for two like charges of charge e and mass 1 unit. There has been an extensive and thorough investigation of the Green function for the non-relativistic Coulomb problem [4]. This explains our choice of propagators and the toy model. (We have deliberately chosen a repulsive interaction as if the quanta 'B' are vector bosons to avoid tachyons [1] and the attendant problem in choosing the boundary conditions for the Green function.) The integral equation can be converted into a differential equation [1]

$$(\nabla_q^2 + \frac{e^2}{q})I(\vec{q}, \vec{q}') = -4\pi\delta^3(\vec{q} - \vec{q}') \quad (2)$$

A solution of this equation with an appropriate asymptotic behaviour is the solution of the integral equation Eq.1. It has been realised that the 'momentum space Green function' of the Coulomb problem

$$G(\vec{p}, \vec{p}') = \int d^3q d^3q' e^{i\vec{q}\cdot\vec{p} - i\vec{q}'\cdot\vec{p}'} I(\vec{q}, \vec{q}') \quad (3)$$

has nice properties. (We emphasise that this terminology of the Coulomb problem is incorrect for our Green function $I(\vec{q}, \vec{q}')$ which is already for the momenta \vec{q}, \vec{q}' and therefore refer to $G(\vec{p}, \vec{p}')$ as MSGF.) This satisfies the integral equation

$$p^2 G(\vec{p}, \vec{p}') + e^2 \int d^3p' \frac{G(\vec{p}', \vec{p}')}{(\vec{p}' - \vec{p})^2} = \delta^3(\vec{p} - \vec{p}') \quad (4)$$

Remarkably the perturbation in e^2 converges for the MSGF [2, 3].

An explicit solution of the Green function of the Coulomb problem has been obtained many times in literature using a variety of techniques. An elegant way is to use the dynamical symmetry provided by the Runge-Lenz vector. In the case of the zero energy Green function relevant to us, the full symmetry group is $E(3)$, the Euclidean group in three dimensions[5]. This suggests the choice of a new variable

$$\vec{\xi} = \frac{e^2 \vec{p}}{p^2} \quad (5)$$

Thus $\vec{\xi}$ is obtained from \vec{p} by inversion in a sphere of radius e^2 . This is a dimensionless variable. Define

$$g(\vec{\xi}, \vec{\xi}') = \frac{(pp')^4}{\pi e^6} G(\vec{p}, \vec{p}') \quad (6)$$

This is also dimensionless. It satisfies the integral equation

$$g(\vec{\xi}, \vec{\xi}') + \int \frac{d^3 \vec{\xi}''}{2\pi^2} \frac{g(\vec{\xi}, \vec{\xi}')}{(\vec{\xi}'' - \vec{\xi}')^2} = \delta^3(\vec{\xi} - \vec{\xi}') \quad (7)$$

This is easily solved by Fourier transform.

$$g(\vec{\xi}, \vec{\xi}') = \int_0^\infty \frac{d^3 k}{2\pi} \frac{k}{k+1} e^{i\vec{k} \cdot (\vec{\xi} - \vec{\xi}')} \quad (8)$$

The leading singularities $g(\vec{\xi}, \vec{\xi}')$ as $\vec{\xi} \rightarrow \vec{\xi}'$ can be obtained by writing

$$\frac{k}{k+1} = 1 - \frac{1}{k} + \frac{1}{k(k+1)} \quad (9)$$

This gives

$$g(\vec{\xi}, \vec{\xi}') = \delta^3(\vec{\xi} - \vec{\xi}') \frac{1}{2\pi^2(\vec{\xi} - \vec{\xi}')^2} + \frac{1}{2\pi^2 t} (\sin(t) ci(t) + \cos(t) si(t)) \quad (10)$$

where $t = |\vec{\xi} - \vec{\xi}'|$ and $ci(t)$, $si(t)$ are the cosine and sine integrals.

The corresponding expansion for MSGF 3 is

$$G(\vec{p}, \vec{p}') = \frac{1}{pp'} \delta^3(\vec{p} - \vec{p}') e^2 \frac{1}{p^2 p'^2} \frac{1}{(\vec{p} - \vec{p}')^2} + (\text{less singular terms}) \quad (11)$$

This is the expansion of the MSGF to $O(e^2)$. Our interest is in $I(\vec{q}, \vec{q}')$, Eq. 1. The first term on RHS of equation 11 gives it a contribution $1/|\vec{q} - \vec{q}'|$ which is the contribution of the tree diagram. The Fourier transform of the second term on RHS of equation is log divergent in the ultraviolet. This matches with the divergence of the 1-box diagram. Thus a perturbation theory fails to be meaningful for our Green function $I(\vec{q}, \vec{q}')$ while it gives a convergent series for the Fourier transform $G(\vec{p}, \vec{p}')$.

We now evaluate $I(\vec{q}, \vec{q}')$ exactly and demonstrate that we get a meaningful result. For this we first perform the angular integrations over the unit vectors \hat{p}, \hat{p}' using the plane wave expansion,

$$e^{i\vec{p} \cdot \vec{q}} = 4\pi \sum i^l j_l(p, q) Y_{lm}^*(\hat{p}) Y_{lm}(\hat{q}) \quad (12)$$

and corresponding expressions for $\exp(-i\vec{p}' \cdot \vec{q}')$, $\exp(-ie^2 \vec{p} \cdot \vec{k}/k^2)$ and $\exp(ie^2 \vec{p}' \cdot \vec{k}/k^2)$. Here the sum is over $l = 0, 1, 2, \dots$ and further for each l , over $m = -l, -l+1, \dots, l-1, l$. Here $j_l(x)$ are the spherical Bessel functions related by $j_l(x) = \sqrt{\pi/2x} J_{l+1/2}(x)$ to the usual Bessel functions. Performing angular integrations using orthogonality

$$\int \frac{d\Omega_p}{4\pi} Y_{lm}^*(\hat{p}) Y_{l'm'}(\hat{p}) = \delta_{ll'} \delta_{mm'} \quad (13)$$

gives

$$I(\vec{q}, \vec{q}') = \int \frac{dp}{p^2} \frac{dp'}{p'^2} \int \frac{dk}{2\pi} \frac{1}{k+1} \sum j_l(pq) j_l\left(\frac{e^2 k}{p}\right) j_l(pq) j_l\left(\frac{e^2 k'}{p'}\right) Y_{lm}^*(\hat{q}) Y_{l'm'}(\hat{q}) \quad (14)$$

We may now do the integrations over p and p' variables using [7]

$$\int \frac{dx}{x^2} J_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right) = \frac{1}{a} J_{2\nu}(2\sqrt{\frac{a}{b}}), a, b > 0, \operatorname{Re} \nu > -\frac{1}{2} \quad (15)$$

We get

$$I(\vec{q}, \vec{q}') = \frac{1}{4\pi e^2} \int_0^\infty \frac{k}{k+1} \sum J_{2l+1}(\sqrt{e^2 k q}) J_{2l+1}(\sqrt{e^2 k' q'}) P_l(\hat{q} \cdot \hat{q}') \quad (16)$$

where we have summed over the m variable for each l to get the Legendre function P_l of the angle between the vectors \vec{q}, \vec{q}' . In order to carry out the final integration over the variable k we use

$$\int_0^\infty dx \frac{x}{x^2 + c^2} J_\nu(ax) J_\nu(bx) = I_\nu(bc) K_\nu(ac), 0 < b < a \quad (17)$$

where I_ν, K_ν are the modified Bessel functions of imaginary argument. We get

$$I(\vec{q}, \vec{q}') = \frac{1}{\sqrt{qq'}} \sum_l (2l+1) I_{2l+1}(\sqrt{e^2 q_<}) K_{2l+1}(\sqrt{e^2 q_>}) P_l(\hat{q} \cdot \hat{q}') \quad (18)$$

where $q_<, q_>$ are respectively the smaller and larger of the momenta q, q' . This gives the 'non-perturbative' evaluation of the sum of ladder diagrams each of which (except the tree diagram) is divergent. Using the leading asymptotic behaviors for large argument,

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \quad (19)$$

$$K_\nu(z) \sim \frac{\pi e^{-z}}{\sqrt{2z}} \quad (20)$$

we can read off the behaviour of the Green function when one of the momenta is much larger than the other:

$$I(\vec{q}, \vec{q}') \sim \frac{e^{-e^2(q_>-q_<)}}{\sqrt{qq'}} \quad (21)$$

We see that the Green function falls off exponentially with momentum, the scale being fixed by the coupling constant e^2 , which has dimension of mass in our 3-Euclidean dimensions. We see that insisting on a perturbative expansion led to the divergences. Defining the Green function through an integral equation gave a result free of divergences. It falls off much faster than the tree approximation which makes the perturbative calculation diverge.

In this Note we argued that the divergences in QFT are due to presumptions regarding the asymptotic behaviours of the Green function of the interacting theory. Calculating Green functions using appropriate integral equation evades the infinities. We used a toy model to explicitly demonstrate our contentions. It is yet to be demonstrated that this can be made into a systematic and a viable tool consistent with requirements such as unitarity and causality in realistic theories. These issues will be addressed elsewhere.

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